Complex Analysis: Final Exam

Aletta Jacobshal 01, Wednesday 27 January 2016, 18:30 – 21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Do not forget to very clearly write your **full name** and **student number** on each answer sheet and on the envelope. Do not seal the ennvelope.
- The exam consists of 6 questions; answer all of them.
- The total number of points is 100 and 10 points are "free".. The exam grade is the total number of points divided by 10.
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (12 points)

(a) (6 points) Verify that the exponential function $f(z) = e^z$ satisfies the Cauchy-Riemann equations.

Solution

Write

$$f(z) = e^{x+iy} = e^x \cos y + ie^x \sin y,$$

and identify

$$u = e^x \cos y, \quad v = e^x \sin y.$$

Then we have that

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Therefore the Cauchy-Riemann equations are satisfied.

(b) (6 points) Show that the Taylor series of the exponential function $f(z) = e^z$ around $z_0 \in \mathbb{C}$ is given by

$$e^z = e^{z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{j!}.$$

What is the domain where the Taylor series of e^z around z_0 converges? [If necessary, you can use without proof the Taylor series of e^z around 0.]

Solution

Write $w = z - z_0$. Then

$$f(z) = e^z = e^{z_0+w} = e^{z_0} e^w = e^{z_0} \sum_{j=0}^{\infty} \frac{w^j}{j!} = e^{z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{j!}$$

Since e^z is entire, the domain of convergence is $\mathbb{C}$.

Question 2 (18 points)

Consider the function

$$f(z) = \frac{e^{iz}}{z^2 - 4}.$$

(a) (6 points) Compute the residue of $f(z)$ at each one of the singularities of $f(z)$.

Solution

The singularities of $f(z)$ are $z = 2$ and $z = -2$, obtained as solutions of $z^2 - 4 = 0$. Each of the singularities is a pole of order 1. Therefore,

$$\text{Res}(f; 2) = \lim_{z \to 2} (z - 2) \frac{e^{iz}}{z^2 - 4} = \lim_{z \to 2} \frac{e^{iz}}{z + 2} = \frac{e^{2i}}{4},$$

and

$$\text{Res}(f; -2) = \lim_{z \to -2} (z + 2) \frac{e^{iz}}{z^2 - 4} = \lim_{z \to -2} \frac{e^{iz}}{z - 2} = -\frac{e^{-2i}}{4}.$$

(b) (12 points) Compute the principal value

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - 4} \, dx.$$

Solution

We have

$$I = \text{pv} \int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 4} \, dx = \lim_{\substack{R \to \infty \\ r_1, r_2 \to 0^+}} \left(\int_{-R}^{-2-r_1} + \int_{-2+r_1}^{2-r_2} + \int_{2+r_2}^{R} \right) \frac{e^{ix}}{x^2 - 4} \, dx.$$

Defining the contour γ_1 as the straight line along the real axis from $-R$ to $-2 - r_1$, the contour γ_2 from $-2 + r_1$ to $2 - r_2$, and γ_3 from $2 + r_2$ to R, we can write

$$I = \lim_{\substack{R \to \infty \\ r_1, r_2 \to 0^+}} \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right) \frac{e^{iz}}{z^2 - 4} \, dz.$$

We define a closed positively oriented contour Γ as

$$\Gamma = \gamma_1 + S_{r_1}^+ + \gamma_2 + S_{r_2}^+ + \gamma_3 + C_R^+,$$

where $S_{r_1}^+$ is the half-circle of radius r_1 and center -2 in the upper half-plane joining $-2-r_1$ to $-2 + r_1$, $S_{r_2}^+$ is the half-circle of radius r_2 and center 2 in the upper half-plane joining $2 - r_2$ to $2 + r_2$, and C_R^+ is the half-circle of radius R and center 0 in the upper half-plane joining R to $-R$.

Then

$$\int_{\Gamma} \frac{e^{iz}}{z^2 - 4} \, dz = 0$$

for all values of R, r_1, r_2 since Γ does not enclose any singularities of the integrand.

Furthermore, we have

$$\lim_{r_1 \to 0^+} \int_{S_{r_1}^+} \frac{e^{iz}}{z^2 - 4} \, dz = -\pi i \, \text{Res}(f; -2) = \frac{\pi i e^{-2i}}{4},$$

and

$$\lim_{r_2\to 0^+} \int_{S_{r_2}^+} \frac{e^{iz}}{z^2-4}\,dz = -\pi i \operatorname{Res}(f;2) = -\frac{\pi i e^{2i}}{4}.$$

For the integral over C_R^+ we have that the coefficient of iz in e^{iz} is positive, and the degree of the denominator in $1/(z^2-4)$ is 2 while the degree of the numerator is 0, and we can apply Jordan's lemma to get

$$\lim_{R\to\infty} \int_{C_R^+} \frac{e^{iz}}{z^2-4}\,dz = 0.$$

Therefore

$$\lim_{\substack{R\to\infty\\ r_1,r_2\to 0^+}} \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{S_{r_1}^+} + \int_{S_{r_2}^+} + \int_{C_R^+}\right) \frac{e^{iz}}{z^2-4}\,dz = \lim_{\substack{R\to\infty\\ r_1,r_2\to 0^+}} \int_\Gamma \frac{e^{iz}}{z^2-4}\,dz = 0,$$

and the left-hand side gives

$$I + \frac{\pi i e^{-2i}}{4} - \frac{\pi i e^{2i}}{4} + 0 = 0.$$

From here

$$I = \frac{\pi i}{4}(e^{2i} - e^{-2i}) = -\frac{\pi}{2}\sin 2.$$

Question 3 (14 points)

Consider the branch $f(z) = e^{\frac{1}{3}\mathcal{L}_0(z)}$ of the cubic root function $z^{\frac{1}{3}}$. The branch $\mathcal{L}_0(z)$ of the logarithm has a branch cut along the positive real axis.

(a) (6 points) Compute $f(-i)$ and $f'(-i)$. Write the results in Cartesian form.

Solution

We have

$$\mathcal{L}_0(-i) = \operatorname{Log}|-i| + i\arg_0(-i) = \frac{3\pi}{2}i.$$

Then

$$f(-i) = e^{\frac{1}{3}\mathcal{L}_0(-i)} = e^{i\pi/2} = i.$$

Moreover,

$$f'(z) = \frac{1}{3}e^{\frac{1}{3}\mathcal{L}_0(z)}\mathcal{L}_0'(z) = \frac{1}{3z}f(z).$$

Then

$$f'(-i) = \frac{1}{3(-i)}f(-i) = -\frac{1}{3}.$$

(b) (8 points) Evaluate the limits $\lim_{\varepsilon\to 0^+} f(x+i\varepsilon)$ and $\lim_{\varepsilon\to 0^+} f(x-i\varepsilon)$ for $x>0$. Express the limits in terms of $\sqrt[3]{x}$, that is, the positive real cubic root of the real number $x>0$.

Solution
We have

$$\lim_{\varepsilon\to 0^+} \mathcal{L}_0(x+i\varepsilon) = \lim_{\varepsilon\to 0^+} \operatorname{Log}|x+i\varepsilon| + i\lim_{\varepsilon\to 0^+} \arg_0(x+i\varepsilon) = \operatorname{Log}|x| = \operatorname{Log} x.$$

We used here that the function $\operatorname{Log}|z|$ is continuous so the first limit is $\operatorname{Log}|x|$ while for $x>0$ and $\varepsilon>0$ the second limit is 0. Then, using the continuity of the exponential,

$$\lim_{\varepsilon\to 0^+} f(x+i\varepsilon) = e^{\frac{1}{3}\lim_{\varepsilon\to 0^+}\mathcal{L}_0(x+i\varepsilon)} = e^{\frac{1}{3}\operatorname{Log} x} = \sqrt[3]{x}.$$

Then we have

$$\lim_{\varepsilon\to 0^+} \mathcal{L}_0(x-i\varepsilon) = \lim_{\varepsilon\to 0^+} \operatorname{Log}|x-i\varepsilon| + i\lim_{\varepsilon\to 0^+} \arg_0(x-i\varepsilon) = \operatorname{Log}|x| + 2\pi i = \operatorname{Log} x + 2\pi i.$$

We used here again that the function $\operatorname{Log}|z|$ is continuous so the first limit is $\operatorname{Log}|x|$ while for $x>0$ and $\varepsilon>0$ the second limit is 2π. Then, using the continuity of the exponential,

$$\lim_{\varepsilon\to 0^+} f(x-i\varepsilon) = e^{\frac{1}{3}\lim_{\varepsilon\to 0^+}\mathcal{L}_0(x-i\varepsilon)} = e^{\frac{1}{3}(\operatorname{Log} x+2\pi i)} = e^{2\pi i/3}\sqrt[3]{x}.$$

Question 4 (14 points)

Consider the function

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}.$$

(a) (4 points) Determine the singularities of $f(z)$ and their type.

Solution
The singularities are $z=1$ and $z=2$ and they are both simple poles (poles of order 1).

(b) (10 points) Compute the Laurent series at 0 of the function in the domain $1<|z|<2$.

Solution
We have

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{j=0}^{\infty}\frac{1}{z^j} = \sum_{j=0}^{\infty}\frac{1}{z^{j+1}} = \sum_{j=1}^{\infty}\frac{1}{z^j} = \sum_{j=-\infty}^{-1} z^j,$$

where in the second step we used the geometric series since $|1/z|<1$.
We also have

$$\frac{2}{2-z} = \frac{1}{1-\frac{z}{2}} = \sum_{j=0}^{\infty}\frac{z^j}{2^j},$$

where in the last step we used the geometric series since $|z/2|<1$.
Therefore the Laurent series in the domain $1<|z|<2$ is

$$f(z) = \sum_{j=-\infty}^{-1} z^j + \sum_{j=0}^{\infty} 2^{-j} z^j,$$

or

$$f(z) = \sum_{j=-\infty}^{\infty} c_j z^j,$$

where $c_j = 1$ for $j\le -1$ and $c_j = 2^{-j}$ for $j\ge 0$.

Question 5 (16 points)

(a) (6 points) Given the function

$$f(z) = \frac{(z-4)(z-1)^2 \sin z}{z^2+1},$$

evaluate the integral

$$\int_C \frac{f'(z)}{f(z)}\, dz,$$

where C is the positively oriented circular contour with $|z| = 2$.

Solution

The Argument Principle gives that under the assumptions in this question we have

$$\int_C \frac{f'(z)}{f(z)}\, dz = 2\pi i[N_0(f) - N_p(f)],$$

where $N_0(f)$ is the number of zeros of $f(z)$ inside C, counting multiplicities, and $N_p(f)$ is the number of poles of $f(z)$ inside C, counting orders.

The function $f(z)$ has zeros at 4, 1 (double zero), and $k\pi$ with $k \in \mathbb{Z}$. The only zeros inside C are 1 (double) and 0. Therefore, $N_0(f) = 3$. The poles are at $\pm i$ and they both lie inside C. Therefore,

$$\int_C \frac{f'(z)}{f(z)}\, dz = 2\pi i[3-2] = 2\pi i.$$

(b) (10 points) Use Rouché's theorem to show that the polynomial $P(z) = \varepsilon z^3 + z^2 + 1$, where $0 < \varepsilon < 3/8$, has exactly 2 roots in the disk $|z| < 2$.

Solution

We apply Rouché's theorem with $f(z) = z^2 + 1$ and $h(z) = \varepsilon z^3$. The function $f(z)$ has exactly two zeros $\pm i$ and they both lie in the disk $|z| < 2$. To conclude that $P(z)$ also has exactly two zeros inside the same disk we must check that $|h(z)| < |f(z)|$ on the circle $|z| = 2$.

For $|z| = 2$ we have

$$|f(z)| = |z^2 + 1| \geq ||z^2| - 1| = ||z|^2 - 1| = 3,$$

and

$$|h(z)| = |\varepsilon z^3| = \varepsilon |z|^3 = 8\varepsilon < 3 \leq |f(z)|.$$

Therefore, $|h(z)| < |f(z)|$ for $|z| = 2$ and applying Rouché's theorem gives the required result.

Question 6 (16 points)

(a) (8 points) Show that

$$\left| \int_C \frac{e^z}{z+1}\, dz \right| \leq 2\pi e^2,$$

where C is the positively oriented circle $|z - 1| = 1$.

Solution

On C we have that $0 \leq x \leq 2$ where $x = \operatorname{Re} z$. It is possible to see this by drawing C or by noticing that $x - 1 = \operatorname{Re}(z) - 1 = \operatorname{Re}(z - 1)$ and we always have $|\operatorname{Re} w| \leq |w|$, so $|x - 1| \leq 1$. Therefore,

$$|e^z| = |e^x e^{iy}| = e^x \leq e^2.$$

Moreover,

$$|z + 1| = |(z - 1) + 2| \geq ||z - 1| - 2| = |1 - 2| = 1.$$

Therefore,

$$\left|\frac{e^z}{z+1}\right| \leq e^2.$$

This means

$$\left|\int_C \frac{e^z}{z+1}\, dz\right| \leq e^2 \ell(C) = 2\pi e^2,$$

where, in the last step, we used that the length of the circle C of radius 1 is 2π.

(b) (8 points) Suppose that $f(z)$ is an analytic function in a domain D and it has no zeros in D. Show that if $|f(z)|$ attains its minimum in D (that is, there exists a point $z_0 \in D$ such that $|f(z_0)| \leq |f(z)|$ for all $z \in D$), then $f(z)$ is constant.

Solution

Consider the function

$$g(z) = \frac{1}{f(z)}.$$

Since $f(z) \neq 0$ for all $z \in D$ we conclude that g is analytic in D.

Furthermore,

$$|g(z_0)| = \frac{1}{|f(z_0)|} \geq \frac{1}{|f(z)|} = |g(z)|,$$

for all $z \in D$. In other words, $|g(z)|$ attains its maximum in D. From the maximum modulus principle this means that $g(z)$ is constant in D and therefore $f(z)$ is also constant in D.

End of the exam (Total: 90 points)